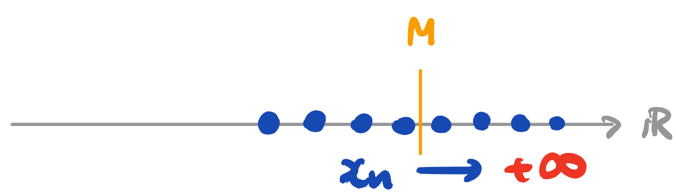
MATH 2050C Lecture 24 (Apr 19)

[Reminder: Last Problem Set 12 due this Friday.]

* Mock Exam this Thursday @ 9:15 AM *

Some "extended" concepts about limits

(i) "Properly divergent seq."



Def$^{\underline{n}}$: $\lim(x_n) = +\infty$

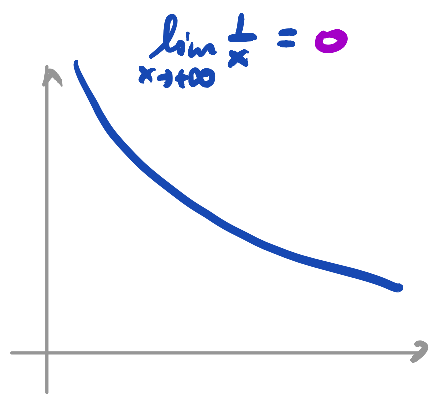
if $\forall M > 0$, $\exists K \in \mathbb{N}$ s.t.

$$x_n > M \qquad \forall n \geq K$$

Remark: These are NOT considered to be convergent seq. since they do NOT satisfy limit theorems.

Eg.) "$\lim(n - n) = 0$"

(ii) "Limit at ∞" for $f : A \to \mathbb{R}$

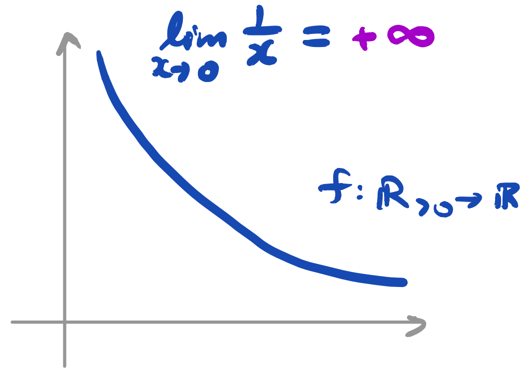


Def$^{\underline{n}}$: $\lim_{x \to +\infty} f(x) = L$

iff $\forall \varepsilon > 0$, $\exists M > 0$ s.t.

$$|f(x) - L| < \varepsilon \quad \text{when } x \in A,\ x > M$$

(iii) "Limit to ∞"



Def$^{\underline{n}}$: $\lim_{x \to c} f(x) = +\infty$

iff $\forall M > 0, \exists \delta > 0$ s.t.

$f(x) > M$ when $x \in A$, $0 < |x - c| < \delta$

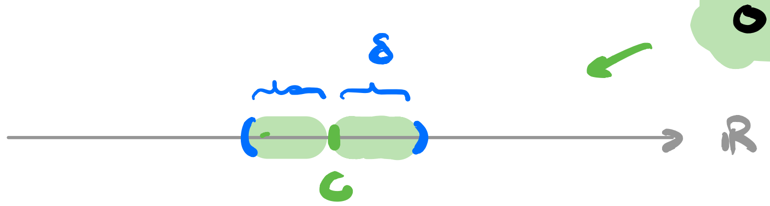
One-sided limits

Recall: $\lim_{x \to c} f(x) = L$ means

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$|f(x) - L| < \varepsilon$ when $x \in A$ and

$0 < |x - c| < \delta$



Sometimes, we only consider $x \approx c$ from one side.

Def$^{\underline{n}}$: $f : A \to \mathbb{R}$, $c \in \mathbb{R}$ is a cluster pt. of $A \cap (c, \infty)$

$\lim_{x \to c^+} f(x) = L$ iff $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$|f(x) - L| < \varepsilon$

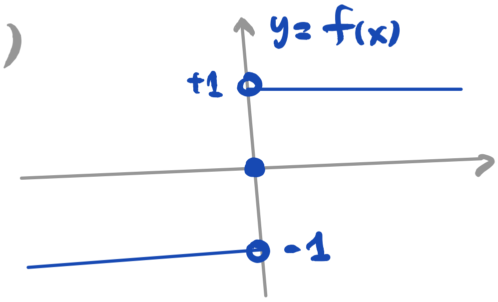
when $x \in A$, $0 < x - c < \delta$

"right hand limit"

Thm: $\lim_{x \to c} f(x) = L \iff \lim_{x \to c^+} f(x) = L = \lim_{x \to c^+} f(x)$

Remark: Many of the "limit theorems" still hold for 1-sided limits.

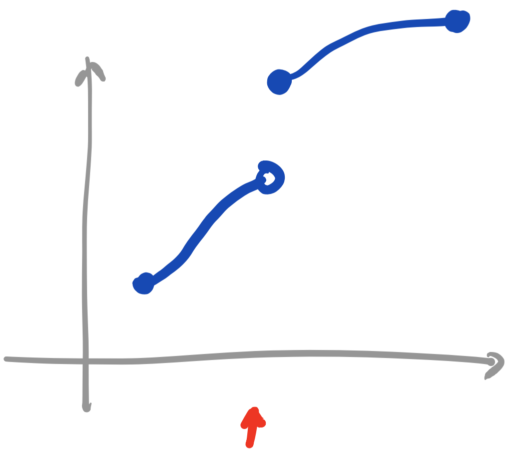
E.g.)



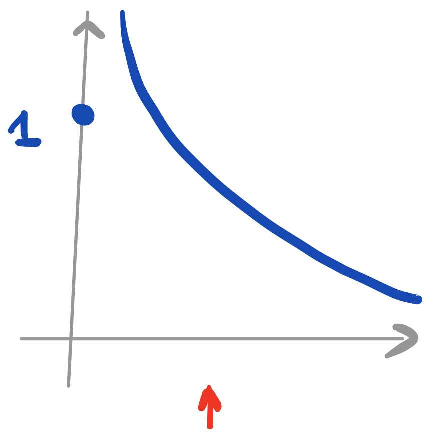
have 1-sided limit at 0 but not 2-sided limit.

Back to continuity

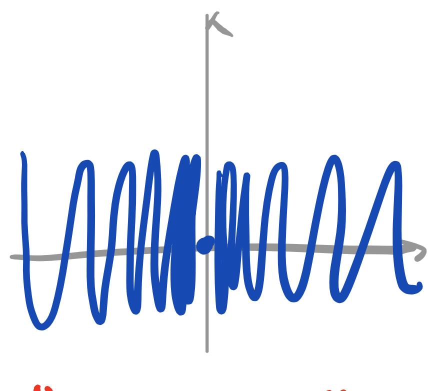
Recall: The following functions are dis-continuous



"jump" discontinuity



"∞ discontinuity"



"Oscillatory" discontinuity

Q: Can we "classify" the types of discontinuities for (simpler) functions $f: A \to \mathbb{R}$?

A: Yes for "monotone" functions.

Def$^{\underline{n}}$: Let $f: A \to \mathbb{R}$. We say

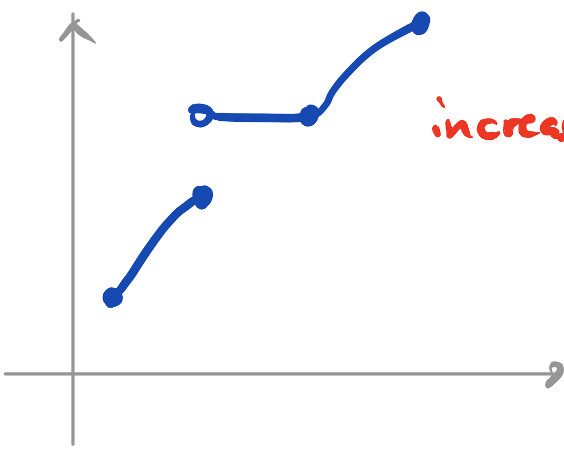
(i) f is (strictly) increasing if

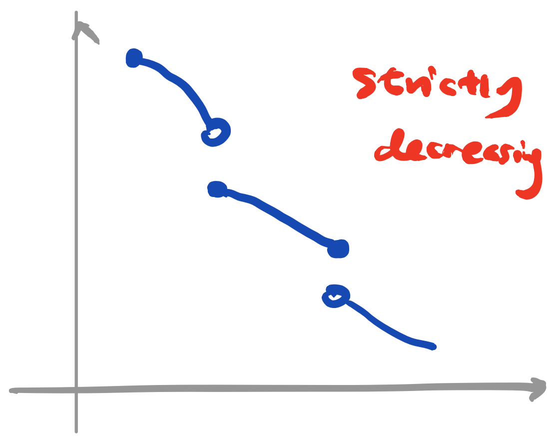
$$x_1, x_2 \in A \ \& \ x_1 \overset{(<)}{\leq} x_2 \Rightarrow f(x_1) \overset{(<)}{\leq} f(x_2)$$

(ii) f is (strictly) decreasing if

$$x_1, x_2 \in A \ \& \ x_1 \overset{(<)}{\leq} x_2 \Rightarrow f(x_1) \overset{(>)}{\geq} f(x_2)$$

(iii) f is (strictly) monotone if it is either (strictly) increasing or decreasing.

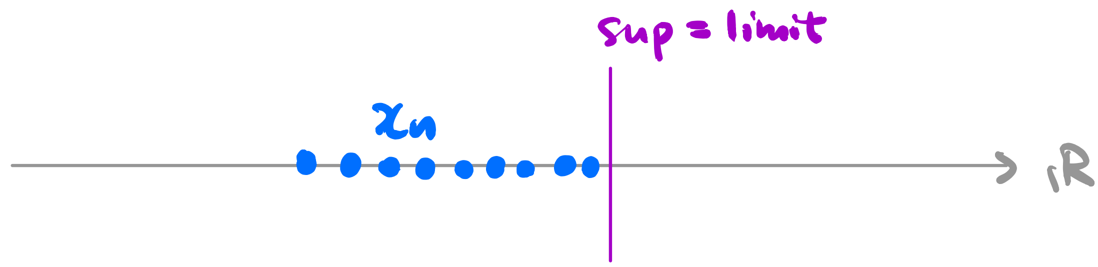




GOAL: Monotone functions defined on $[a.b]$ only have "jump" discontinuities.

Recall: MCT for seq:

(x_n) increasing & bdd above $\Rightarrow$ $\lim(x_n) = \sup\{x_n \mid n \in \mathbb{N}\}$

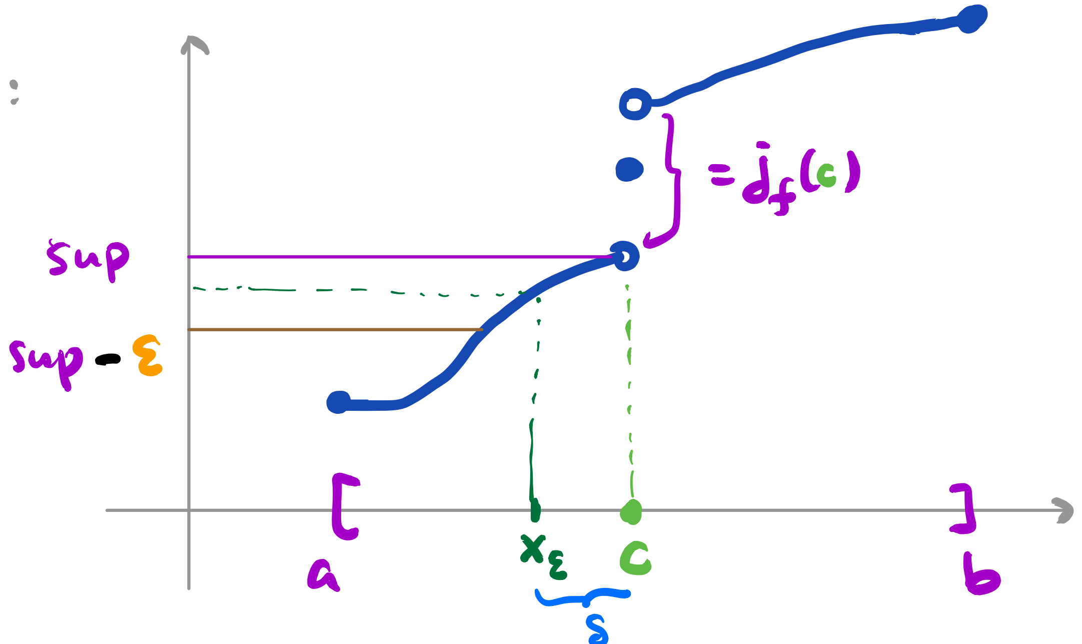


Thm: Let $f: [a.b] \to \mathbb{R}$ be an increasing fcn.

For any $c \in (a,b)$, we have

$$\lim_{x \to c^-} f(x) = \sup_{x \in [a,c)} f(x) \quad \& \quad \lim_{x \to c^+} f(x) = \inf_{x \in (c,b]} f(x)$$

Picture:



Proof: **Claim:** $\lim_{x\to c^-} f(x) = \sup_{x\in[a,c)} f(x)$ ← exists

∵ bdd above by $f(c)$

Let $\varepsilon > 0$ be fixed but arbitrary.

By def^n of supremum, $\exists\, x_\varepsilon \in [a,c)$ s.t

$$\sup_{x\in[a,c)} f(x) - \varepsilon < f(x_\varepsilon) \le \sup_{x\in[a,c)} f(x)$$

Choose $\delta := c - x_\varepsilon > 0$. Then, since f is increasing, we have $\forall\, x \in [a,c)$ s.t $0 < c - x < \delta$

$\Rightarrow x_\varepsilon < x < c$ and thus

$$\sup_{x\in[a,c)} f(x) - \varepsilon < f(x_\varepsilon) \le f(x) \le \sup_{x\in[a,c)} f(x)$$

So. $\lim_{x\to c^+} f(x) = \sup_{x\in[a,c)} f(x)$

$\blacksquare$

Remark: Under the same assumption as in Thm.

f is cts at $c\in[a,b]$ $\iff$ $\lim_{x\to c^+} f(x) = f(c) = \lim_{x\to c^-} f(x)$

limits exist by Thm.

Denote: The "jump" of f at c as

$$j_f(c) := \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x)$$

Note: $j_f(c) \geq 0$ and "=" $\iff$ f cts at c

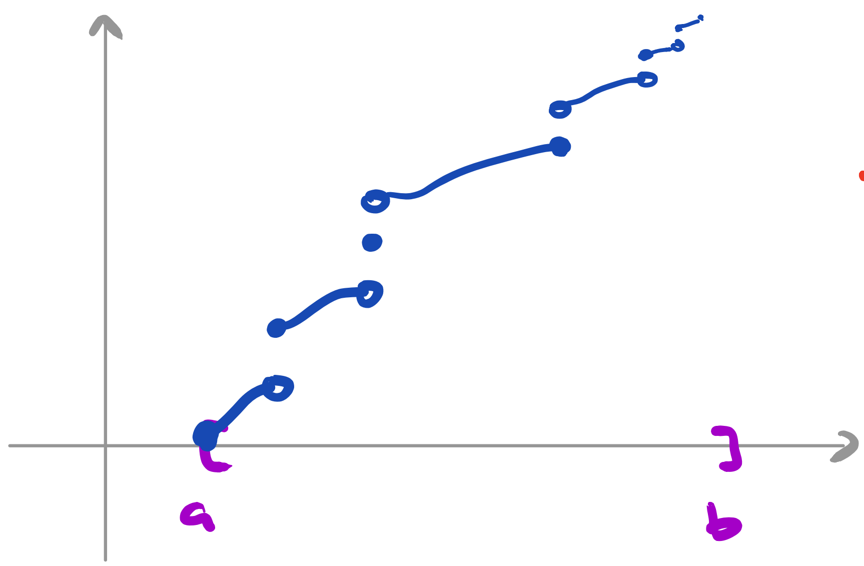
Thm: Let $f: [a,b] \to \mathbb{R}$ be an increasing fcn.

THEN, the set of discontinuities

$$\mathcal{D} := \{ c \in [a,b] : f \text{ discts at } c \}$$

is at most countable.

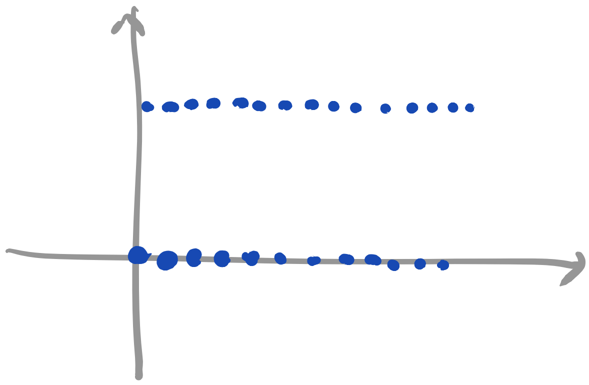
Picture: "worst case scenario" for increasing fcn.



$\exists$ at most countably many "jumps".

Remark: Without monotonicity, it can be much worse.

E.g.)



$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

is discts EVERYWHERE in $[0,1]$.

Proof: Note that:

$$\mathcal{D} = \{ c \in (a,b) : j_f(c) > 0 \}$$

Observe that $j_f(c) \leq f(b) - f(a)$.

Consider the following subsets:

$$\mathcal{D}_1 := \{ c \in (a,b) : j_f(c) \geq f(b) - f(a) \} \qquad \# \mathcal{D}_1 \leq 1$$

$$\mathcal{D}_2 := \{ c \in (a,b) : j_f(c) \geq \frac{f(b) - f(a)}{2} \} \qquad \# \mathcal{D}_2 \leq 2$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\mathcal{D}_k := \{ c \in (a,b) : j_f(c) \geq \frac{f(b) - f(a)}{k} \} \qquad \# \mathcal{D}_k \leq k$$

Then, $\mathcal{D} = \bigcup_{k=1}^{\infty} \mathcal{D}_k$ is at most countable.

$\blacksquare$

Existence of inverse (for monotone fcn)

Given $f : [a.b] \to \mathbb{R}$ cts. then

Extreme Value Thm

$\Rightarrow$ $m := \inf_{x \in [a,b]} f(x)$ & $M := \sup_{x \in [a,b]} f(x)$

are achieved. So, $m, M \in \text{Range}(f)$

Intermediate Value Thm

$$\Rightarrow \quad f([a.b]) = [m, M]$$

i.e. cts function takes a closed & bdd interval to another closed & bdd interval.

Q: When does the inverse $f^{-1} : [m, M] \to [a.b]$ exist?

Thm: If $f : [a.b] \to \mathbb{R}$ is strictly increasing and cts, then $f^{-1} : [m, M] \to [a.b]$ exists, and still strictly increasing and cts.

"Idea of Proof":

(by cts of f)

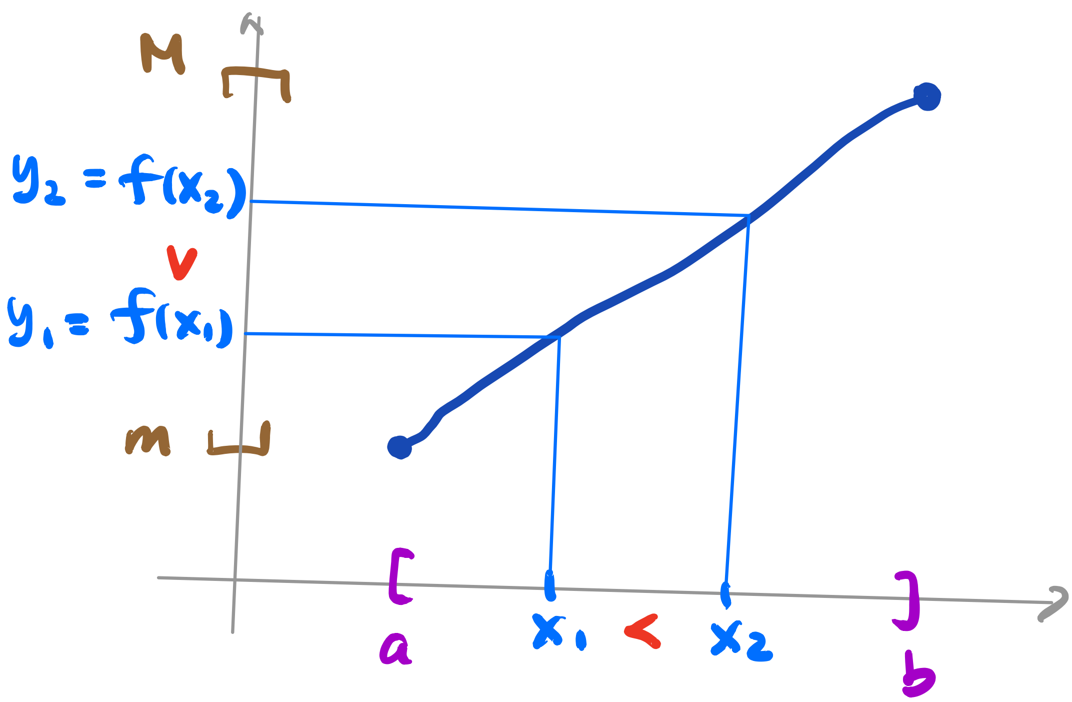
EVT, IVT $\Rightarrow$ $f : [a,b] \to [m,M]$ is onto

f str. increasing $\Rightarrow$ also 1-1.

Therefore, f^{-1} exists.

Claim 1: $f^{-1} : [m,M] \to [a,b]$ str. increasing.

Claim 2: f^{-1} is cts.



$\square$